# ON THE CLOSENESS OF APPROACH OF COMPLEX RATIONAL FRACTIONS TO A COMPLEX IRRATIONAL NUMBER*

BY

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Dirichlet showed that if $\omega$ is a real irrational, the inequality

$$\left| \omega - \frac{p}{q} \right| < \frac{k}{q^2}$$

is satisfied by an infinite number of real rational fractions, $p/q$, when $k = 1$. Later Hermite gave a method, based on binary quadratic forms, of constructing an infinite suite of fractions satisfying the inequality when $k = 1/\sqrt{3}$. The problem of the minimum value of $k$ was solved by Hurwitz[†], who showed that if $k = 1/\sqrt{5}$ an infinite number of fractions satisfy the inequality, whereas if $k < 1/\sqrt{5}$ there is an $\omega$ in every interval of the real axis for which the inequality holds for only a finite number of rational fractions. Proofs of Hurwitz' theorem have been given by Borel[‡], by Humbert[§] and by the present author[||].

In the present paper we propose to investigate the analogous problem in the complex domain. Let $\omega$ be any complex irrational number and consider the inequality

(1)
$$\left| \omega - \frac{p}{q} \right| < \frac{k}{q\bar{q}},$$

where $k$ is real, $p/q$ is a complex rational fraction (i. e., $p$ and $q$ are each of the form $m + in$, where $m$ and $n$ are real integers), and $\bar{q}$ is the conjugate imaginary of $q$.

Hermite[¶] has demonstrated, again by the use of quadratic forms, the existence of an infinite suite satisfying the inequality when $k = 1/\sqrt{2}$. However, the problem of the least value of $k$, such that for any $\omega$ there

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† Mathematische Annalen, vol. 39 (1891), pp. 279–284.

‡ Journal de Mathématiques, ser. 5, vol. 9 (1903), pp. 329 ff.

§ Journal de Mathématiques, ser. 7, vol. 2 (1916), pp. 155–167.

|| Proceedings of the Edinburgh Mathematical Society, vol. 35, Part 2, Session 1916–1917.

¶ Journal für Mathematik, vol. 47 (1854), pp. 357–366. Hurwitz, in Acta Mathematica, vol. 11 (1887), pp. 187–200, gives an infinite suite satisfying the inequality when $k = 1$.

is always an infinite number of fractions satisfying the inequality, has not been solved hitherto. The difficulty has been that the early methods used in the real case, those of Hurwitz and Borel, employ continued fractions and make use of properties not possessed by any satisfactory extension of ordinary continued fractions to the case of complex numbers. The geometric methods used by the author are not subject to this limitation.

We shall prove the following

THEOREM. *If $k = 1/\sqrt{3}$ there is an infinite number of rational fractions satisfying the inequality*

$$(1) \qquad\qquad \left| \omega - \frac{p}{q} \right| < \frac{k}{q\bar{q}}.$$

*If $k < 1/\sqrt{3}$ there exists a set of irrational numbers, everywhere dense in the complex plane, for each of which the inequality (1) is satisfied by only a finite number of rational fractions.*

**The geometric problem. $L$-lines and $S$-spheres.** Visualizing the Argand diagram on which the complex numbers are represented as a horizontal plane, we shall be concerned with geometrical constructions in the three-dimensional space lying on one side of this plane, say above it. Through the point $\omega$ under consideration let a line, $L$, be drawn perpendicular to the complex plane. At each rational point, $p/q$ (in its lowest terms), let a sphere, $S$, be constructed, tangent to the complex plane at that point, of radius $1/2\,h\,q\,\bar{q}$, and lying in the upper half-space.

If $L$ intersects the $S$-sphere corresponding to $p/q$ the distance between $\omega$ and $p/q$ is less than the radius, or

$$(2) \qquad\qquad \left| \omega - \frac{p}{q} \right| < \frac{1}{2\,h\,q\bar{q}},$$

otherwise the inequality does not hold. We must show then that when $h = \frac{1}{2}\sqrt{3}$, $L$ intersects an infinite number of these spheres and that when $h > \frac{1}{2}\sqrt{3}$ the $L$-lines constructed through certain of the irrationals intersect only a finite number of spheres.*

**The group of Picard.** The spheres which we have just defined are connected with the group of linear transformations

$$(3) \qquad\qquad z' = \frac{az+b}{cz+d}, \qquad ad - bc = 1,$$

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* It is clearly unnecessary to consider fractions not in their lowest terms. If $p/q$ fails to satisfy an inequality of the type (2) an equal fraction with a larger absolute $q$ will fail to satisfy it; if $p/q$ satisfies the inequality only a finite number of equal fractions will satisfy it.

where $a$, $b$, $c$, $d$ are complex integers, in the following manner. If the transformations of the group be defined as space transformations according to the method of Poincaré[*] the upper half-space is transformed into itself. If we add to the spheres of the preceding section the plane $\zeta = h$ ($\zeta$ being measured along an axis through the origin perpendicular to the complex plane) the resulting set of spheres is invariant under the transformations of the group.[†] The plane $\zeta = h$ is the $S$-sphere of the point $\infty$.

**Division of the half-space into pentahedra.** Of prime importance in the study of the group is the fundamental pentahedron, discovered by Bianchi.[‡] Write $z = \xi + i\eta$. The fundamental pentahedron is the portion of space lying above the unit sphere with center at the origin, $\xi^2 + \eta^2 + \zeta^2 = 1$, and bounded by the four planes $\xi = \pm \frac{1}{2}$, $\eta = \pm \frac{1}{2}$. If this solid be inverted in its faces and each new pentahedron be inverted in its faces, and so on ad infinitum, the whole upper half-space is filled up without overlapping. This set of pentahedra has the property of invariance; that is, each transformation of the group carries each pentahedron into some other. Furthermore, there exists a transformation carrying a given pentahedron into a specified one; in particular any one can be carried into the fundamental pentahedron by a suitable transformation.

The invariant set of pentahedra and the invariant set of $S$-spheres will play a fundamental part in the proof. One further fact will be used: The segment of an $L$-line bounded by a point in the upper half-space and by the irrational point $\omega$ in the complex plane intersects an infinite number of pentahedra.

**Proof of the first part of the theorem.** Let $h = \frac{1}{2}\sqrt{3}$. Let us suppose that for a given irrational $\omega$ the corresponding $L$-line intersects only a finite number of $S$-spheres. Then, in the neighborhood of $\omega$, $L$ passes successively through an infinite number of pentahedra and remains exterior to, or at most tangent to, all $S$-spheres. We shall show that this is impossible.

Let $D$ be a pentahedron through which $L$ passes. Make a transformation of the group of Picard carrying $D$ into the fundamental pentahedron. Then, since circles are carried into circles and angles are preserved, $L$ is carried into a semi-circle orthogonal to the complex plane and passing through the fundamental pentahedron.

[*] Acta Mathematica, vol. 3 (1884), pp. 49-92. Poincaré bases his extension on the well known fact that a linear transformation in the complex plane is equivalent to an even number of inversions in circles. By making inversions in spheres having these circles as equators a space transformation results. Certain properties are immediate: the transformations are conformal; spheres are carried into spheres; and circles are carried into circles.

[†] For a full discussion of the geometry of the group of Picard see a paper by the present author, these Transactions, vol. 19 (1918), pp. 1-42.

[‡] Mathematische Annalen, vol. 38 (1891), pp. 313-333.

We shall now investigate whether we can so place a semi-circle, $C$, that it shall be orthogonal to the $z$-plane, that it shall pass through the fundamental pentahedron, and that it shall intersect none of the $S$-spheres that lie in the neighborhood. The points of the fundamental pentahedron nearest the $z$-plane (the vertices) have the $\zeta$-coördinate $\frac{1}{2}\sqrt{2}$. This fact, together with the requirement that $C$ shall not penetrate the region above the plane $\zeta = \frac{1}{2}\sqrt{3}$, gives for the radius, $r$, of $C$ the following bounds:

$$\frac{1}{2}\sqrt{2} \leqq r \leqq \frac{1}{2}\sqrt{3}.$$

Consider now the $S$-spheres of the integral points. These are tangent to the complex plane at the integral points and have the radius $1/\sqrt{3}$. They are cut by a plane $\zeta = $ const. in a set of circles whose centers lie vertically above the integral points. Consider the $S$-sphere of the point $0/1$; viz.,

$$\xi^2 + \eta^2 + \zeta^2 - \frac{2}{\sqrt{3}}\zeta = 0.$$

Setting $\zeta = \frac{1}{2}\sqrt{3}$ and again $\zeta = \sqrt{3}/6$ in this equation we have in each case

$$\xi^2 + \eta^2 = \frac{1}{4}.$$

Since the radius is $\frac{1}{2}$ we see that each of these planes cuts the $S$-spheres of the integral points in a set of tangent circles [Fig. 1]. Horizontal planes lying between these two planes cut the spheres in larger circles.
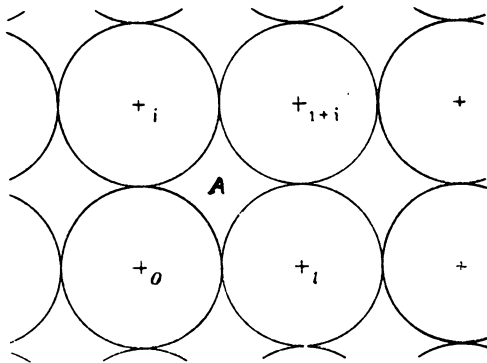


Fig. 1.

Let $K_1$, $K_2$ be the intersections of $C$ with the plane $\zeta = \sqrt{3}/6$. If $C$ is to remain exterior to the spheres in question, $K_1$ and $K_2$ must lie in the regions, such as $A$ in Fig. 1, exterior to the circles. Now $K_1$ and $K_2$ cannot lie in the same region $A$; for the greatest length that can be laid down in $A$ is 1, whereas the chord $K_1 K_2$ at a distance $\sqrt{3}/6$ from the center of a circle of radius not less than $\frac{1}{2}\sqrt{2}$ is easily found to be not less than $\sqrt{15}/3$, or 1.29.

There remains the possibility that $K_1$ and $K_2$ lie in different regions, the arc $K_1 K_2$ passing above certain of the $S$-spheres under consideration. Now considering Fig. 1 as the intersection of the plane $\zeta = \frac{1}{2}\sqrt{3}$ with

the $S$-spheres, we see that adjacent $S$-spheres have a common horizontal
tangent line lying in the plane $\zeta = \frac{1}{2}\sqrt{3}$ at the points where the circles
of intersection are tangent. The only position which $C$ can have in order
not to penetrate into the interior of one of these spheres or pass above
the plane $\zeta = \frac{1}{2}\sqrt{3}$ is to touch two spheres and the plane at the point
where they have a common horizontal tangent.

The foregoing restricts $C$ to one of four positions in the faces of the
fundamental pentahedron. On account of symmetry it will suffice to con-
sider one case: $C$ lies in the plane $\xi = \frac{1}{2}$, has the radius $\frac{1}{2}\sqrt{2}$, and
the center $\xi = \frac{1}{2}$, $\eta = \zeta = 0$.

$C$ meets the $z$-plane in the points

$$z_1 = \frac{1}{2} + \frac{1}{2}i\sqrt{3}, \qquad z_2 = \frac{1}{2} - \frac{1}{2}i\sqrt{3}.$$

Now both these points are irrational, whereas the transform of an $L$-line
meets the $z$-plane in one rational point; namely, the transform of $\infty$. For
if $z = \infty$ in (3) we have $z' = a/c$, a rational. Consequently this semi-
circle cannot be the transform of an $L$-line.

We have proved that every transform of an $L$-line which passes through
the fundamental pentahedron has a segment either above the plane $\zeta = \frac{1}{2}\sqrt{3}$
or within an $S$-sphere corresponding to an integral point, and furthermore
that this enclosed segment lies above the plane $\zeta = \sqrt{3}/6$. We can con-
struct a region above the latter plane within which we are certain that
such a segment lies; for instance, the region above the plane $\zeta = \sqrt{3}/6$
and enclosed by the cylinder $\xi^2 + \eta^2 = 25$. Let $N$ be the number of penta-
hedra extending within this region. It is known from the geometry of the
pentahedral division that $N$ is finite. The number of pentahedra in the
region through which the semicircular transform of $L$ passes is less than $N$.

Carrying these results back to the original pentahedron $D$ we can state
that of some $N$ successive pentahedra, including $D$, through which $L$ passes,
there is a segment of $L$ lying within an $S$-sphere. It follows that $L$ cannot pass
successively through an infinite number of pentahedra and remain exterior to,
or at most tangent to, all $S$-spheres. This proves the first part of the theorem.

**Proof of the second part of the theorem.** Consider the semi-circle $C$
just discussed. We shall show that for its terminus, $\omega = \frac{1}{2} + \frac{1}{2}i\sqrt{3}$, the
inequality (2) holds for only a finite number of rational fractions when
$h > \frac{1}{2}\sqrt{3}$.

The proof will require a careful study of the situation of $C$ with reference
to the $S$-spheres when $h = \frac{1}{2}\sqrt{3}$. We shall show that, with this value
of $h$, $C$ is tangent to an infinite number of $S$-spheres but penetrates into
the interior of none.

The irrationals $\frac{1}{2} + \frac{1}{2} i \sqrt{3}$ and $\frac{1}{2} - \frac{1}{2} i \sqrt{3}$ are the roots of the equation

$$z^2 - z + 1 = 0.$$

This equation can be written in the form

$$z = \frac{(2-i)z + 2i}{-2iz + 2 + i},$$

which shows that $z_1$ and $z_2$ are the fixed points of the transformation

$T$: $$z' = \frac{(2-i)z + 2i}{-2iz + 2 + i}.$$

Since

$$ad - bc = (2-i)(2+i) + (2i)^2 = 1,$$

it follows that $T$ is a transformation of the group of Picard.

Since $a + d = 4$ the transformation is of the type known as hyperbolic (the condition being that $a + d$ be real and $|a + d| > 2$). It is characteristic of this type of transformation that all circular arcs joining fixed points are invariant under the transformation. $C$ is such an invariant arc.

Furthermore, if $P$ is a point on $C$ and $P'$ is its transform when the transformation $T$ is made, then by repeating the transformation $T$ and its inverse the transforms of the arc $PP'$ cover the semi-circle $C$ completely without overlapping. We shall presently choose a convenient point $P$, after which it will suffice to determine the situation of the arc $PP'$ with reference to $S$-spheres in order to know the situation of the whole semi-circle.

We shall begin by considering the $S$-spheres of the numbers $-i$ and $1/(1 + i)$. Their equations are

$S_1(-i)$: $$\xi^2 + (\eta + 1)^2 + \left(\zeta - \frac{1}{\sqrt{3}}\right)^2 = \left(\frac{1}{\sqrt{3}}\right)^2,$$

$S_2\left(\dfrac{1}{1+i}\right)$: $$\left(\xi - \frac{1}{2}\right)^2 + \left(\eta + \frac{1}{2}\right)^2 + \left(\zeta - \frac{1}{2\sqrt{3}}\right)^2 = \left(\frac{1}{2\sqrt{3}}\right)^2.$$

The intersections of these spheres with the plane $\xi = \frac{1}{2}$, in which $C$ lies, are the circles

$C_1$: $$(\eta + 1)^2 + \left(\zeta - \frac{1}{\sqrt{3}}\right)^2 = \frac{1}{12},$$

$C_2$: $$\left(\eta + \frac{1}{2}\right)^2 + \left(\zeta - \frac{1}{2\sqrt{3}}\right)^2 = \frac{1}{12}.$$

These circles in the plane $\xi = \frac{1}{2}$ are shown drawn to scale in Fig. 2.

We shall now show that $C$ is tangent to these circles. The equation of $C$ in the plane $\xi = \frac{1}{2}$ is

$$C: \qquad \eta^2 + \zeta^2 = \left(\frac{1}{2} \sqrt{3}\right)^2 = \frac{3}{4}.$$

From the proportionality of the coördinates of the centers of $C_1$ and $C_2$,

$$-1 : \frac{1}{\sqrt{3}} = -\frac{1}{2} : \frac{1}{2\sqrt{3}},$$

we see that the centers lie on a line,

$$\eta = -\sqrt{3}\,\zeta,$$

through the center $(0,0)$ of $C$. This line cuts $C$ in a point, $P$, whose coördinates are found to be $\eta = -\frac{3}{4}$, $\zeta = \frac{1}{4}\sqrt{3}$. We verify easily that $P$ lies on both $C_1$ and $C_2$. Since the circles pass through a point on their line of centers they are mutually tangent at the point.

We note further that the $S$-sphere of the point $1 - i$ cuts the plane $\xi = \frac{1}{2}$ in $C_1$, since the sphere is the reflection of $S_1$ in that plane. We have shown then that $C$, which was constructed so as to be tangent to the $S$-spheres of $0$, $1$, and $\infty$, is tangent also to the $S$-spheres of $-i$, $1 - i$, and $1/(1 + i)$.
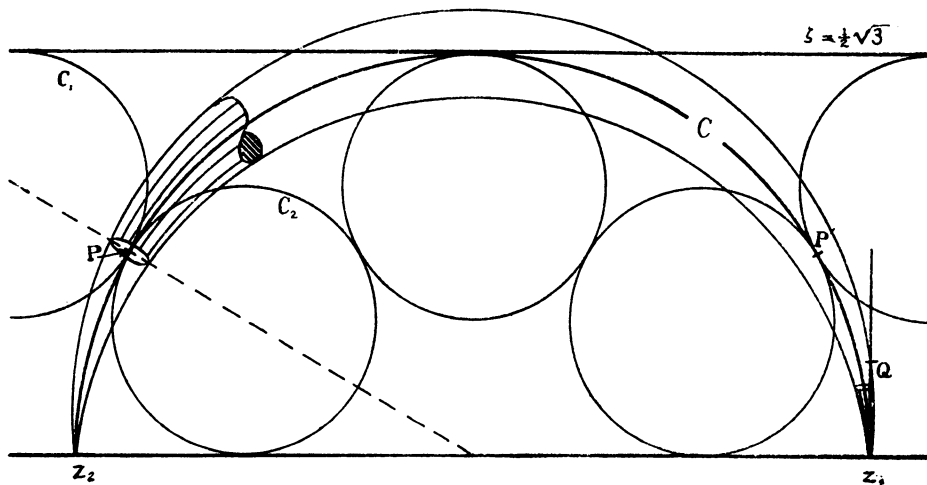


Fig. 2.

If now we make a reflection in the vertical plane through the real axis, $\eta = 0$, with respect to which the system of pentahedra, the system

of $S$-spheres, and the semi-circle $C$ are symmetrical, the spheres corresponding to the points $-i$, $1-i$, and $1/(1+i)$ are carried into the spheres corresponding to $i$, $1+i$, and $1/(1-i)$ respectively. Hence the spheres corresponding to these latter points touch $C$ at a point $P'$, whose coördinates in the plane $\xi = \frac{1}{2}$ are $\eta = \frac{3}{4}$, $\zeta = \frac{1}{4}\sqrt{3}$.

Now let us make the transformation $T$. The points $-i$, $1-i$, and $1/(1+i)$ are carried, as we find by substituting in the equation of the transformation, into the points $i$, $1+i$, and $1/(1-i)$ respectively. Since $S$-spheres go into $S$-spheres and since $C$ is invariant, it follows that $P$ is transformed into the point common to $C$ and to the $S$-spheres of these latter three points; that is, into $P'$.

The nine $S$-spheres tangent to $C$ along the arc $PP'$ are transformed by repetitions of $T$ and its inverse into an infinite number of $S$-spheres tangent to $C$.

In order to prove that no point of $C$ is interior to an $S$-sphere it suffices to prove that no point of $PP'$ is interior to an $S$-sphere. Now in order that an $S$-sphere should contain a point of $PP'$ it is necessary that its diameter be greater than the $\zeta$-coördinate of $P$; that is,

$$\frac{2}{\sqrt{3}\,q\bar{q}} > \frac{1}{4}\sqrt{3}, \quad \text{or} \quad q\bar{q} < \frac{8}{3}.$$

Now $q\bar{q} \geqq 4$ except for the values $q = 1$ and $q = 1+i$ (or these values multiplied by $-1$ or $\pm i$). The former are the integral points; the latter are the points $m + ni + \frac{1}{2}(1+i)$.

Again an $S$-sphere cannot contain any point of $C$ unless the distance from the rational point to which the sphere belongs to the nearest point of the segment $z_1 z_2$ is less than the radius. The radii of the two classes of spheres just mentioned are $1/\sqrt{3}$ and $\sqrt{3}/6$. There are no points of these two classes, other than $0$, $1$, $i$, $-i$, $1+i$, $1-i$, $\frac{1}{2}(1+i)$, and $\frac{1}{2}(1-i)$, which have already been considered, lying within the larger of these two distances. The nearest is $\frac{1}{2}(1+3i)$ (or the similarly situated point $\frac{1}{2}(1-3i)$) whose distance from $z_1$ is $|\frac{1}{2}(1+3i)-\frac{1}{2}(1+i\sqrt{3})|$, or $\frac{1}{2}(3-\sqrt{3})$, which is greater than $1/\sqrt{3}$.

We have shown that the arc $PP'$ is tangent to certain $S$-spheres but penetrates to the interior of none. It follows that the whole semi-circle $C$ has no points interior to an $S$-sphere but that it touches an infinite number of $S$-spheres.

Now let $h$ be greater than $\frac{1}{2}\sqrt{3}$. Then the plane $\zeta = h$ lies above the plane $\zeta = \frac{1}{2}\sqrt{3}$, and each $S$-sphere formed with this value of $h$ lies within the $S$-sphere formed with the value of $h = \frac{1}{2}\sqrt{3}$. It follows that $C$ now touches no $S$-spheres throughout its entire length.

**The invariant tube $\Gamma$.** We shall now construct a sort of tube, $\Gamma$, enclosing $C$ as follows: With $P$ as center construct a circle, $K$, in a plane normal to $C$, and generate a surface by the motion of a circle through $z_1, z_2$, and a point which moves around the circumference of $K$. We shall take $K$ small enough that the tube from the neighborhood of $P$ to that of $P'$ be exterior to $S$-spheres.

The tube is invariant under the transformation $T$. For, the arc through $z_1$ and $z_2$ which generates it is, in each position, an invariant arc.

The plane area enclosed by $K$ is transformed by $T$ into a spherical surface through $P'$ bounded by a circle $K'$. If we apply repeatedly the transformation $T$ and its inverse to the portion of the solid tube bounded by these two sections at $P$ and $P'$, its transforms fill up without overlapping the whole interior of $\Gamma$. It follows that any point within $\Gamma$ is exterior to all $S$-spheres.

The tube terminates at $z_1$ in a conical point. Now take $\omega = z_1$, and construct the $L$-line there. This line, being normal to the $z$-plane, lies entirely within $\Gamma$ in the neighborhood of $z_1$ and passes out of the tube at some point $Q$ above $z_1$. Between $z_1$ and $Q$ the line is exterior to all $S$-spheres. Above $Q$ it can intersect only a finite number of $S$-spheres. Hence the inequality (2) is satisfied by only a finite number of rational fractions.

If we make a transformation of the type (3), $z_1$ is carried into an irrational $z'$. $\Gamma$ is carried into a tube exterior to all $S$-spheres. The transformed tube has a conical point at $z'$; and the $L$-line erected at $z'$ intersects only a finite number of $S$-spheres.

It follows that all points into which $z_1$ can be carried by transformations of the group of Picard are irrationals for which the inequality (2) is satisfied by only a finite number of rational fractions. But the transforms of this point, as of any point in the $z$-plane, are everywhere dense in the $z$-plane. This establishes the second part of the theorem.

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